# Parameter resolution bounds that depend on sample size

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It is currently common practice in theoretical ocean acoustics to derive parameter resolution bounds for a monochromatic measurement of the temporally fluctuating field received by a hydrophone array. However, a monochromatic measurement corresponds to a single random sample. In applied ocean acoustics, single samples are seldom if ever used for parameter estimation because the associated error can be unnecessarily large. Instead estimates are derived from ensemble averages such as the sample covariance. To bridge the gap between these two approaches, the Fisher Information for the sample covariance is shown to be equal to the number of independent and stationary samples times the Fisher Information for a single sample. Therefore, *there are no practical limits on parameter resolution* if (1) the bound for a single sample is finite, which is generally the case of interest, (2) a sufficiently large population of independent samples can be found. The parameter resolution issue then becomes one of determining the maximum number of such samples. This number is set by physical variables that do not appear in the monochromatic or instantaneous measurement. A means of determining this number from the temporal coherence of the received field and the measurement time is presented.

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## INTRODUCTION

Recently an inconsistency has emerged between theoretical and applied ocean acoustics. A number of parameter resolution bounds have appeared in the literature. Many of these bounds have analytic forms that do not allow for the possibility of ensemble averaging, including some bounds given by the present author, and therefore predict unrealistically poor resolution for a given set of parameters.[1–5] This is in direct contrast to current practice in applied ocean acoustics where ensemble averaging is widely used to lower the variance in parameter estimation.

In particular, the complex sample covariance of a sensor array is generally the measurement from which parameters are estimated in applications of narrow-band matched field processing and beamforming.[6] The sample covariance is obtained by ensemble averaging instantaneous measurements of a sensor array's spatial covariance for a fixed narrow-band about a given frequency. While the averaging is usually done by sampling in the time domain, the result is equivalent to averaging the same number of independent frequency components across the narrow-band of the measurement. As an estimator of the true covariance, the sample covariance has obvious advantages over an instantaneous measurement because its variance can be linearly reduced with the number of independent and stationary samples in the estimate.

However, many theoretical parameter resolution bounds for underwater acoustics collapse the finite bandwidth of the sample covariance to a single-frequency or monochromatic measure.[1–5] But a monochromatic measurement is statistically similar to an instantaneous one in the sense that each constitutes only a single independent sample of a temporally fluctuating quantity. This is because the time-bandwidth product of a monochromatic measurement, like that of an instantaneous one, is by definition unity. Clearly, the variance of a parameter estimate derived from a single sample may be unnecessarily large if additional independent samples are available. But it is implicit in ocean-acoustic practice that such additional samples are typically available because the very evidence that is necessary to establish the validity of a statistical approach must be obtained from the observation of far more than a single random sample of the measurement variable. This same abundance of samples can then be used to reduce the variance in a parameter estimate.

To bridge the gap between theory and practice, the Fisher information for a measurement of the complex sample covariance is shown to be equal to the number of independent and stationary samples available times the Fisher information for a single sample. Therefore, *there are no practical limits on parameter resolution* if (1) the bound for a single sample is finite, which is generally the case of interest, (2) a sufficiently large population of such samples can be found.

The parameter resolution issue then becomes one of determining the maximum number of independent and identically distributed samples available. However, this number is set by physical variables that do not appear in the instantaneous or monochromatic measurement, and consequently, is not accounted for in the monochromatic bounds of the present ocean-acoustics literature.[1–5] To remedy this situation, a means of determining the maximum number of independent samples available in a given stationary measurement period is presented.

It is notable that in their matched field treatments of broadband signals, Song[7,8] as well as Fawcett and Maranda[9] have shown that the optimal position resolution of a target in a waveguide can generally improve by increasing the duration of the received signal, if other parameters such as the signal bandwidth and noise characteristics are held fixed. This is consistent with well-established radar range-estimation bounds derived for targets in free space.[10] Apparently, the implication of these results has not been fully re-

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alized in ocean-acoustic modeling of narrow-band signals. Therefore, it is the aim of the present paper to explicitly show how optimal parameter resolution is increased by averaging independent fluctuations of the measurement variable.

In Sec. I, a general expression is derived for the Fisher information matrix contained in a measurement of the complex sample covariance. This is done without specific knowledge of the probability distribution of the measurement. Two practical cases are examined in Secs. II and III, respectively. First, the entire received field is assumed to be fully randomized. Then, the signal is allowed to have a nonfluctuating component.

In the first case, it is assumed that the instantaneous fields received by the sensor array are circular complex Gaussian random (CCGR) variables[11] so that both the signal and noise components fluctuate over temporal samples. This is a standard assumption implicit in the derivation of many ocean acoustics parameter bounds,[1-5] but is also well founded in a variety of other fields such as optics and radar.[11,12] It is additionally assumed that measurements across the array may be correlated at any instant, but all discrete samples are independent and stationary over time. The Fisher information matrix for the sample covariance is then found to equal the product of the Fisher information matrix for an instantaneous measurement of the field across the array and the number of independent and stationary samples. Consequently, the Fisher information matrix of Refs. 1–5 is found to only be valid for the special case of a single monochromatic measurement.

Previously derived monochromatic parameter resolution bounds,[1-5] therefore, are reduced in direct proportion to the available number of independent and stationary samples when finite bandwidth is allowed. An expression for this number is given in terms of the temporal coherence of the received field and the measurement time. Appropriately substituting this expression into an analytic formula recently derived by Lee et al.[13] for the exact distribution of a discrete-sample-covariance element yields a new approximate probability distribution for an element of the continuous-time sample covariance. This approximate distribution is the complex counterpart to Rice's distribution for mean-square noise current[14] and Mandel's distribution for the intensity fluctuations of a photon beam.[15] In optical applications, it can be used to describe the statistical properties of finite-time-averaged mutual intensity[11] under the assumption of cross-spectral purity.[16]

In the second case, it is assumed that the field received by the sensor array contains a deterministic signal component amid an additive CCGR component. It is shown that the Fisher information matrix for this sample covariance is different from the fully random case but is directly proportional to the number of independent samples available, as anticipated. Optimal parameter resolution, therefore, can again be significantly improved by averaging identically distributed instantaneous covariance samples. The number of independent samples available is again given in terms of the temporal coherence of the random component of the received field and the measurement time.

Two illustrative examples that have a significant impact on the interpretation of published literature are given in Sec. IV. These are for matched field tomography of internal waves, and target detection using ambient noise as a source of opportunity.

## I. FISHER INFORMATION FOR THE COMPLEX SAMPLE COVARIANCE

The random vector $\mathbf{\Phi}[n]$ contains the instantaneous complex fields $\phi_i[n]$ measured by sensors $i = 1,2,3,\ldots,N_\Phi$ at discrete time $n$. To permit spatial coherence across the array at any instant $n$, it is assumed that any sample $\phi_i[n]$ may be correlated with any $\phi_j[n]$. However, all field samples with differing discrete-time indexes are assumed to be independent. For the intended application in interferometry, all fields are assumed to occupy a narrow band about the same carrier frequency such that the bandwidth is much less than the propagation speed divided by the array aperture. More precise constraints on the bandwidth for application in matched field processing are discussed in Sec. IV. A specific extension to broadband applications is provided for a fully randomized field in Sec. II.

The complex sample covariance is then defined as

$$\mathbf{S}[N] = \frac{1}{N} \sum_{n=1}^{N} \mathbf{\Phi}[n] \mathbf{\Phi}^H[n], \tag{1}$$

where $\mathbf{S}[N]$ is a Hermitian matrix containing the elemental measurement statistics $S_{ij}[N]$, for $i,j = 1,2,3,\ldots,N_\Phi$, where $^H$ denotes complex conjugate transpose. Here, the instantaneous mutual intensity[11] samples $\mathbf{\Phi}[n]\mathbf{\Phi}^H[n]$ are assumed to be identically distributed over time index $n$. Practical scenarios for which this assumption is valid are investigated in Secs. II and III.

The complex vector $\mathbf{W}$ of dimension $N_\Phi^2$ is formed by stacking the columns of $\mathbf{S}$, so that the elements of $\mathbf{W}$ are given by $W_{ij} = S_{ij}[N]$, for $i,j = 1,2,3,\ldots,N_\Phi$, where the double subscripted vector element follows notation discussed in the Appendix for the Kronecker product.[17] It is noteworthy that in this context $\mathbf{W}^T = \mathbf{W}^*$. The statistics of $\mathbf{W}$ are characterized by mean vector $\mathbf{M} = \langle \mathbf{W} \rangle$, with elements $M_{ij}$, and covariance matrix $\mathbf{C}(N)$, with elements

$$C_{ij,kl}(N) = \langle (W_{ij} - M_{ij})(W_{kl} - M_{kl})^* \rangle,$$

where $i,j,k,l = 1,2,3,\ldots,N_\Phi$. The dependence of the covariance $\mathbf{C}(N)$ on the number of independent and stationary samples $N$ is explicitly indicated for future reference.

The vector $\mathbf{g}$ contains the parameters $g_k$ for $k = 1,2,3,\ldots,N_g$. These parameters are assumed to be nonstochastic but presumably can be estimated from the stochastic data $\mathbf{W}$.

While the general distribution for $\mathbf{W}$ remains unknown and may be unrealizable in compact form,[18] it is assumed to have a Gaussian asymptote for large $N$. This Gaussian asymptote can be written as a conditional probability, for $\mathbf{W}$ given $\mathbf{g}$, by the relation

$$P(\mathbf{W}|\mathbf{g}) \approx \frac{\exp\{-\frac{1}{2}[\mathbf{W} - \mathbf{M}(\mathbf{g})]^T \mathbf{C}^{-1}(\mathbf{g};N)[\mathbf{W} - \mathbf{M}(\mathbf{g})]\}}{((2\pi)^{N_\Phi^2}|\mathbf{C}(\mathbf{g};N)|)^{1/2}}. \tag{2}$$

2852    J. Acoust. Soc. Am., Vol. 99, No. 5, May 1996

Nicholas C. Makris: Parameter resolution bounds    2852

To be consistent with the Kronecker product and the fact that both $\mathbf{W}$ and $\mathbf{M}$ have double subscripts and contain all the elements of corresponding Hermitian matrices, the algebra is akin to vector[19,20] rather than complex vector[18,20] algebra. Specifically, the transpose in the exponent is just the transpose and not the Hermitian transpose, and the factors 2 and $\frac{1}{2}$, as well as the power $\frac{1}{2}$ are appropriately designated in Eq. (2).

For estimation of the parameters $\mathbf{g}$, the elements of the Fisher information matrix $\mathbf{J}(\mathbf{g})$ are defined by[19]

$$J_{ij}(\mathbf{g}) = -E\left[ \frac{\partial^2}{\partial g_i \partial g_j} \ln P(\mathbf{W}|\mathbf{g}) \right]. \quad (3)$$

Since the exact form of the distribution for $P(\mathbf{W}|\mathbf{g})$ is unknown, it seems that the Fisher information matrix for the sample covariance cannot be determined directly from Eq. (3). However, there is an alternative approach.

It is well established that if $\mathbf{g}$ is a function of the $N_a$ parameters in vector $\mathbf{a}$ that have a known Fisher information matrix $\mathbf{J}(\mathbf{a})$, the Fisher information matrix $\mathbf{J}(\mathbf{g})$ can be determined from[19,20]

$$\mathbf{J}(\mathbf{g}(\mathbf{a})) = \left[ \frac{\partial \mathbf{g}(\mathbf{a})}{\partial \mathbf{a}} \mathbf{J}^{-1}(\mathbf{a}) \frac{\partial \mathbf{g}(\mathbf{a})^T}{\partial \mathbf{a}} \right]^{-1}. \quad (4)$$

Suppose the parameter $\mathbf{a}$ is the mean of $\mathbf{W}$ such that $\mathbf{a} = \mathbf{M}$. Then because $P(\mathbf{W}|\mathbf{a})$ is asymptotically Gaussian for sufficiently large $N$, it is also well established that the inverse Fisher information matrix $\mathbf{J}^{-1}(\mathbf{M})$, in an estimate of the mean $\mathbf{M}$, is equal to the covariance matrix $\mathbf{C}(N)$ for all $N \geq 1$.[20] Therefore, substituting $\mathbf{M}$ for $\mathbf{a}$ and then $\mathbf{C}(N)$ for $\mathbf{J}^{-1}(\mathbf{M})$ in Eq. (4) yields the Fisher information matrix $\mathbf{J}(\mathbf{g}(\mathbf{M}))$ for all $N \geq 1$.

This same result can also be obtained by loosely following Fisher's use of the central limit theorem in his initial derivation[21] of what later became known as Fisher information. First note that $\mathbf{C}(N) = \mathbf{C}(1)/N$ must hold due to the independence and stationarity of the $N$ samples in the sample covariance of Eq. (1). Substituting the Gaussian asymptote of $P(\mathbf{W}|\mathbf{g})$ into the right-hand side of Eq. (3) then yields the well-known result[20]

$$\frac{1}{2} \text{Tr} \left\{ \mathbf{C}^{-1}(1) \frac{\partial \mathbf{C}(1)}{\partial g_i} \mathbf{C}^{-1}(1) \frac{\partial \mathbf{C}(1)}{\partial g_j} \right\} + N \frac{\partial \mathbf{M}^T}{\partial g_i} \mathbf{C}^{-1}(1) \frac{\partial \mathbf{M}}{\partial g_j}, \quad (5)$$

which in the present case approaches $J_{ij}(\mathbf{g})$ as $N$ increases. Clearly, only the term involving partial derivatives of $\mathbf{M}$ with respect to the parameters to be estimated survives as $N$ becomes sufficiently large. But the resulting asymptotic expression for $J_{ij}(\mathbf{g})$ is valid for all $N \geq 1$ because the Fisher information for the sum of $N$ independent and identically distributed joint measurements equals $N$ times the Fisher information for a single such measurement.[22] For example, this is why $\mathbf{J}(\mathbf{M}) = N\mathbf{C}^{-1}(1)$ must hold.

The general Fisher information matrix $\mathbf{J}(\mathbf{g})$ for the sample covariance vector $\mathbf{W}$ can then be written element by element as

$$J_{ij}(\mathbf{g}) = N \frac{\partial \mathbf{M}^T}{\partial g_i} \mathbf{C}^{-1}(1) \frac{\partial \mathbf{M}}{\partial g_j}. \quad (6)$$

Again, this expression is exact for the general case of $N \geq 1$. Further, application of Eq. (6) is not limited to the Hermitian sample covariance measurements of Eq. (1), but is valid for any ensemble average of $N$ independent and identically distributed samples that has asymptotically Gaussian statistics for $N$ sufficiently large, as follows from the referenced works of Fisher and Rao for real measurement vectors.

The direct proportionality to sample size exhibited in Eq. (6) may seem obvious because Fisher information is additive for independent samples[19,20,22] and is not lost when such independent identically distributed samples are combined[22] as in Eq. (1). However, even with these facts, the preceding analysis side-stepped the difficulty in first obtaining an expression for the joint distribution of the instantaneous covariance elements, which may not have an analytically realizable form, and then substituting this expression into Eq. (3).

Equation (6) represents the Fisher information for an arbitrary population of $N$ independent samples, which may be collected over time or space, so long as the statistics are ergodic.[11] If the statistics are not ergodic over the entire population, the expression must be modified so as to add an additional term of the form of Eq. (6) for each ergodic subpopulation with its own mean, covariance and sample size. If the measurements are restricted to temporal samples that are not stationary, the expression must again be modified so as to add an additional term of the form of Eq. (6) for each stationary subpopulation. The merit of this approach becomes questionable if the parameters to be estimated vary significantly during what was presumed to be a stationary measurement. For simplicity, from hereon the stochastic component of the field received at a given hydrophone is assumed to be stationary in the sense that its temporal coherence depends only upon the time difference between any two measurements.

It is useful to make the dependence on $N$ explicit in the Fisher information matrix $\mathbf{J}(\mathbf{g};N)$ so that the additive property of the information in independent identically distributed samples is clearly exhibited via $\mathbf{J}(\mathbf{g};N) = N\mathbf{J}(\mathbf{g};1)$. Then the Cramer–Rao lower bound[19,20] is given as

$$E[(\hat{g}_i - g_i)^2] \geq [\mathbf{J}^{-1}(\mathbf{g};N)]_{ii} = \frac{1}{N} [\mathbf{J}^{-1}(\mathbf{g};1)]_{ii}, \quad (7)$$

which unambiguously shows that the limiting mean-square estimation error for any unbiased estimate $\hat{g}_i$ of parameter $g_i$ can be made arbitrarily small if (1) a sufficiently large population $N$ of independent and identically distributed samples can be found, and (2) the $N = 1$ single-sample bound is finite.

In summary, Eqs. (6) and (7) bound the minimum error in estimating parameters from the average of $N$ independent and identically distributed measurement samples, and exhibit a well-known inverse dependence on sample size $N$.[19–22] Some specific cases are now considered.

## II. THE RECEIVED FIELD IS FULLY RANDOMIZED

### A. Fisher information for the complex sample covariance

It is assumed that the instantaneous field samples $\phi_i[n]$ are CCGR variables regardless of the relative amplitude between the signal and noise components. It is again assumed that the field can be spatially correlated across the array at any instant, and that all field measurements with differing discrete-time indexes are independent.

The expectation value of the sample covariance has elements given by

$$M_{ij} = E\left[\frac{1}{N}\sum_{n=1}^{N} \phi_i[n]\phi_j^*[n]\right] = \langle \phi_i[n]\phi_j^*[n]\rangle, \qquad (8)$$

where $M_{ij}$ is independent of time. By use of CCGR moment factoring,[11] the covariance of the sample covariance is found to have the elements

$$C_{qr,st}(N) = E\left[\frac{1}{N^2}\sum_{n=1}^{N}\sum_{m=1}^{N} (\phi_q[n]\phi_r^*[n])(\phi_s[m]\phi_t^*[m])^*\right]$$
$$- M_{qr}M_{st}^*$$
$$= \frac{1}{N} M_{qs}M_{rt}^*. \qquad (9)$$

The Fisher information matrix of Eq. (6) can be written as

$$J_{ij}(\mathbf{g}) = N\sum_{q=1}^{N_\Phi}\sum_{r=1}^{N_\Phi}\sum_{s=1}^{N_\Phi}\sum_{t=1}^{N_\Phi} \frac{\partial[\mathbf{M}^T]_{qr}}{\partial g_i}[\mathbf{C}^{-1}(1)]_{qr,st}\frac{\partial[\mathbf{M}]_{st}}{\partial g_j}. \qquad (10)$$

Following the Appendix, the identity $[\mathbf{C}^{-1}(1)]_{qr,st} = [\mathbf{M}^{-1}]_{qs}[\mathbf{M}^{*-1}]_{rt}$ is employed. By also noting that $[\mathbf{M}^T]_{qr} = [\mathbf{M}]_{rq}$ and $[\mathbf{M}^{*-1}]_{rt} = [\mathbf{M}^{-1}]_{tr}$, the Fisher information matrix can be written as

$$J_{ij}(\mathbf{g}) = N\sum_{q=1}^{N_\Phi}\sum_{r=1}^{N_\Phi}\sum_{s=1}^{N_\Phi}\sum_{t=1}^{N_\Phi} [\mathbf{M}^{-1}]_{tr}\frac{\partial[\mathbf{M}]_{rq}}{\partial g_i}[\mathbf{M}^{-1}]_{qs}\frac{\partial[\mathbf{M}]_{st}}{\partial g_j}. \qquad (11)$$

Returning again to more familiar double subscripted matrix notation, let the expected covariance of an instantaneous sample of the field received across the array be denoted by the matrix $\mathbf{K} = \langle\mathbf{\Phi}[n]\mathbf{\Phi}^H[n]\rangle$, which when converted to a double subscripted vector by stacking its columns yields $\mathbf{M}$. For example, the relationship between the elements of $\mathbf{K}$ and $\mathbf{M}$ is given by $K_{ij} = M_{ij}$. Equation (11) can then be written as

$$J_{ij}(\mathbf{g};N) = N\ \mathrm{tr}\left\{\mathbf{K}^{-1}\frac{\partial\mathbf{K}}{\partial g_i}\mathbf{K}^{-1}\frac{\partial\mathbf{K}}{\partial g_j}\right\}. \qquad (12)$$

Because the signal and noise are assumed to be fully randomized, absolute phase information is lost between independent temporal samples. In this case, there is no difference between the Fisher information for a single instantaneous sample $\mathbf{\Phi}[n]$ of the field across the array and that for a single instantaneous sample $\mathbf{\Phi}[n]\mathbf{\Phi}^H[n]$ of the corresponding mutual intensities. Therefore Eq. (12) also defines the Fisher information matrix for $N$ independent and identically distributed joint measurements of the instantaneous field $\mathbf{\Phi}[n]$ across the array. This last observation provides a useful confirmation of the foregoing derivation because it enables Eq. (12) to be immediately obtained from the well-known form of the Fisher information matrix for a fully randomized zero-mean complex Gaussian measurement vector[20] and the additive property of Fisher information for independent measurements.[19,20,22]

The Fisher information matrix used in Refs. 1–5 for a monochromatic measurement of the temporally fluctuating field received by a hydrophone array is only valid for the special case when $N = 1$ and the measurement is of sufficiently narrow bandwidth. As noted in the introduction, however, a single sample is the worst possible sample population. Therefore, a Cramer–Rao lower bound derived for $N = 1$ has questionable relevance as a ''fundamental'' bound on parameter resolution because it gives the *best* possible parameter resolution for the *worst* possible number of samples. The ramifications of Eq. (12) on specific results in the literature are fleshed out in Sec. IV. An evaluation of the number of independent samples available in a given time period under the CCGR field assumption is presented next.

### B. Sample size as a function of time and coherence

An expression for the maximum number of independent samples available in a stationary measurement period is now derived. This is given in terms of the temporal coherence of the received field and the measurement time. The general approach is analogous to that used by Rice[14] and Mandel[15] for related problems. In loose terms, the concept is to determine the number of times the received field is expected to fluctuate independently during the given measurement period. This is achieved by inspection of the signal-to-noise ratio (SNR) of the measurement. Here the SNR is defined as the squared-mean to variance ratio. For the discretely sampled case, the SNR of a sample covariance element is

$$\mathrm{SNR}\{S_{ij}[N]\} = \frac{|M_{ij}|^2}{C_{ij,ij}(N)} = N\frac{|M_{ij}|^2}{M_{ii}M_{jj}}, \qquad (13)$$

where $M_{ii}$ is positive semidefinite and equal to the expected intensity at sensor $i$. Here the number of independent samples $N$ is equal to the SNR for a diagonal element of the sample covariance, such that $\mathrm{SNR}\{S_{ii}[N]\} = N$. This is because $S_{ii}[1]$ has an expectation value that equals its standard deviation under the CCGR field assumption, and all $N$ samples are independent and identically distributed.

Analogously, the number of independent samples available in a continuous time measurement of $S_{ii}$ is given by its SNR. To show this, the sample covariance of Eq. (1) can be equivalently written as a continuous temporal average

$$\mathbf{S}(T) = \frac{1}{T}\int_{-T/2}^{T/2}\mathbf{\Phi}(t)\mathbf{\Phi}^H(t)\,dt, \qquad (14)$$

where the received fields $\mathbf{\Phi}(t)$ are again assumed to be narrow band. For the continuous case, the SNR of $S_{ij}(T)$ is defined as

$$\text{SNR}\{S_{ij}(T)\} = \frac{\langle R_{ij}(T)\rangle^2 + \langle I_{ij}(T)\rangle^2}{\sigma_{R_{ij}(T)}^2 + \sigma_{I_{ij}(T)}^2}, \tag{15}$$

where $R_{ij}(T) = \text{Re}\{S_{ij}(T)\}$, $I_{ij}(T) = \text{Im}\{S_{ij}(T)\}$, $\sigma_{R_{ij}(T)}^2$ is the variance of $R_{ij}(T)$, and $\sigma_{I_{ij}(T)}^2$ is the variance of $I_{ij}(T)$. It is not difficult to show that $\langle R_{ij}(T)\rangle = \text{Re}\{M_{ij}\}$ and $\langle I_{ij}(T)\rangle = \text{Im}\{M_{ij}\}$.

Expressions for the variances can also be obtained, but with more difficulty. First, it is useful to employ some definitions from statistical optics. The complex degree of coherence[11] is defined as

$$\gamma_{ij}(\tau) = \langle \phi_i(t+\tau)\phi_j^*(t)\rangle / (M_{ii}M_{jj})^{1/2}, \tag{16}$$

and the complex coherence factor[11] is defined as

$$\nu_{ij} = \gamma_{ij}(0) = M_{ij}/(M_{ii}M_{jj})^{1/2}. \tag{17}$$

Because the complex degree of coherence $\gamma_{ij}(\tau)$ only depends on the time difference $\tau$, the fields $\phi_i(t)$ must be stationary over the measurement period $T$ for this analysis to be valid.

By defining the normalized cross-spectral density $\mathscr{S}_{ij}(f)$ as the Fourier transform of the complex degree of coherence $\gamma_{ij}(\tau)$, and $Q_{ij}(f) = (M_{ii}M_{jj})^{1/2}\mathscr{S}_{ij}(f)$ as the unnormalized cross-spectral density which is the Fourier transform of the mutual coherence function $\langle \phi_i(t+\tau)\phi_j^*(t)\rangle$, the expectation value of the sample covariance can be expressed as

$$M_{ij} = \int_{-\infty}^{\infty} Q_{ij}(f)df. \tag{18}$$

This spectral representation is useful for computing the Fisher information matrix of Eq. (12) with state-of-the-art ocean-acoustic propagation models.

For illustrative purposes, it is now assumed that the measurements across the array are cross-spectrally pure.[16] The mathematical expression of cross-spectral purity is $\gamma_{ij}(\tau) = \nu_{ij}\gamma(\tau)$, where $\gamma(\tau) = \gamma_{ii}(\tau)$, for all sensors $i$ and $j$. This means that the normalized cross-spectral density measured between any two spatially separated sensors is the same as the normalized spectral density measured at any sensor in the array. While this condition is most easily satisfied for narrow-band measurements,[16] which are most commonly used in ocean-acoustic interferometry,[1–9] it is also applicable to broadband processes.

With these definitions, expressions for the variances of the real and imaginary parts of the sample covariance elements can be obtained by extending some results presented by Goodman,[11] in his discussion of the properties of mutual intensity, to more the general forms

$$\sigma_{R_{ij}(T)}^2 = \frac{M_{ii}M_{jj}}{2\mu} (2|\text{Re}(\nu_{ij})|^2 - |\nu_{ij}|^2 + 1), \tag{19a}$$

$$\sigma_{I_{ij}(T)}^2 = \frac{M_{ii}M_{jj}}{2\mu} (2|\text{Im}(\nu_{ij})|^2 - |\nu_{ij}|^2 + 1), \tag{19b}$$

where $R_{ij}(T)$ and $I_{ij}(T)$ are found to be uncorrelated. Here, $\mu$ is defined by

$$\mu = \left[ \frac{1}{T} \int_{-\infty}^{\infty} \Delta\left(\frac{\tau}{T}\right) |\gamma(\tau)|^2 d\tau \right]^{-1}, \tag{20}$$

and the triangle function is defined as

$$\Delta(\tau) = 1 - |\tau|, \quad \text{for } |\tau| \leq 1, \tag{21a}$$

$$\Delta(\tau) = 0, \quad \text{elsewhere.} \tag{21b}$$

In terms of the spectral density $\mathscr{S}(f)$, which is the Fourier transform of $\gamma(\tau)$, a useful spectral representation for $\mu$ is given by

$$\mu = \left[ \frac{1}{T^2} \int \int_{-\infty}^{\infty} \mathscr{S}(f)\mathscr{S}^*(f') \left( \frac{\sin(\pi T(f-f'))}{\pi(f-f')} \right)^2 df' \, df \right]^{-1}. \tag{22}$$

By appropriate substitution of the means and variances given above, Eq. (15) yields the desired SNR for the continuous measurement case

$$\text{SNR}\{S_{ij}(T)\} = \mu |\nu_{ij}|^2. \tag{23}$$

Therefore, the number of independent samples available in continuous measurement time $T$ is $\mu$, where $\mu$ need not be discrete but must be greater than or equal to one, as is evident by inspection of Eqs. (20) and (22). It is important to realize that if the assumption of cross-spectral purity cannot be made, the number of independent samples $\text{SNR}\{S_{ii}(T)\}$ would not necessarily be identical across all sensors $i$ of the array, but would be given by Eqs. (20) or (22) with $\gamma(\tau)$ and $\mathscr{S}(f)$ replaced by $\gamma_{ii}(\tau)$ and $\mathscr{S}_{ii}(f)$, respectively.

The continuous sample size $\mu$ can also be interpreted as the time-bandwidth product of the cross-spectrally pure field received by the array.[11,23] For example, by defining the coherence time scale as

$$\tau_c = \int_{-\infty}^{\infty} |\gamma(\tau)|^2 \, d\tau = \int_{-\infty}^{\infty} |\mathscr{S}(f)|^2 \, df, \tag{24}$$

where $1/\tau_c$ measures the bandwidth of field fluctuations over an infinite time window, either Eq. (20) or (22) can be used to show that as $T$ becomes much greater than $\tau_c$, $\mu$ approaches $T/\tau_c$. Specifically, the triangle function approaches a constant in Eq. (20) and the sinc-squared function approaches a delta function in Eq. (22). Therefore, the effect of the finite measurement window on the bandwidth of the received field $B = \mu/T$ becomes negligible in this limit so that the time-bandwidth product $\mu$ grows linearly with measurement time $T$.

In the opposite extreme, inspection of Eq. (20) or (22) indicates that as $\tau_c$ becomes much greater than $T$, the time-bandwidth product $\mu$ approaches unity and can no longer be approximated by $T/\tau_c$. This is because the coherence time scale of Eq. (24) is independent of the temporal window associated with a specific measurement. Here, the complex degree of coherence behaves as a constant in Eq. (20) and the spectral density behaves as a delta function in Eq. (22). This limiting case describes both the monochromatic and the in-

stantaneous measurement, for which the number of independent samples must be unity, as is well known.[11] With regard to the introduction of this paper, the foregoing asymptotic analysis provides a mathematical basis for many of the points raised concerning monochromatic measures and sample size. Specifically, the monochromatic measurement is the subset of instantaneous measurements for which the spectral density $\mathscr{S}(f)$ not only behaves as a delta function in Eq. (22) because $\tau_c \gg T$, but actually is a delta function $\mathscr{S}(f) = \delta(f - f_0)$ in the sense that there is no $T$ for which $\tau_c \gg T$ is not true. Therefore, it is important to realize that a narrow-band measurement is not necessarily monochromatic, but a monochromatic model is often sufficient to approximate the interferometric processes involved in an instantaneous narrow-band measurement. However, an instantaneous broadband measurement must generally be described by its full spectrum $\mathscr{S}(f)$ in any modeling effort even though its time-bandwidth product is unity.

It is useful to now consider the influence that the SNR of $S_{ij}(T)$ has on estimators found in ocean-acoustic beamforming and matched field processing, since these rely upon intersensor coherence. For such estimators, Eq. (23) implies that as the intersensor measurements become less correlated under decreasing $|\nu_{ij}|$, longer averaging times are necessary to achieve the same resolution for a given parameter. For practical considerations, the temporal augmentation necessary for an off-diagonal element at $i,j$ to attain the SNR that a diagonal element achieved in time $T$ is expected to be $(T/|\nu_{ij}|^2 - T)$, when $T \gg \tau_c$.

Finally, the minimum sample size necessary, but not always sufficient, to achieve a fixed mean-square error tolerance in a parameter estimate can be equated with the ratio of the Cramer–Rao lower bound for a single sample to the error threshold. This minimum value then sets practical requirements on the bandwidth and measurement time necessary to attain the desired resolution. Illustrative examples along this line are given in Sec. IV.

## C. Approximate distributions for finite-time-averaged mutual intensity

Under the CCGR field assumption, the approximate distribution for an element of the continuous-time sample covariance $S_{ij}(T)$ can be obtained by combining the results of the last section with an exact distribution recently derived by Lee et al.[13] for an element of the discrete sample covariance $S_{ij}[N]$. This approximate distribution is the complex counterpart to Rice's distribution for mean-square noise current[14] and Mandel's distribution for the intensity fluctuations of a photon beam.[15] In acoustic, optical, and radar applications, it can be used to describe the statistical properties of finite-time-averaged mutual intensity[11] under the assumption of cross-spectral purity. The idea is to follow the analogous derivations of Rice and Mandel by substituting the continuous-time SNR $\mu$ for the number of independent samples $N$ in the exact distribution for the discrete time measurement $S_{ij}[N]$.

While the immediate application of Lee et al.[13] was for polarimetric synthetic aperture radar imaging under the CCGR field assumption, their exact distributions are clearly useful in underwater acoustic interferometry. Specifically, in terms of its complex magnitude $\alpha_{ij}$ and phase $\psi_{ij}$, the discrete sample covariance element $S_{ij}[N]$ is distributed according to[13]

$$P_{\alpha_{ij}, \psi_{ij}}(\alpha, \psi) = \frac{2N^{N+1}\alpha^N \exp\left\{\dfrac{2N\alpha\rho_{ij}\cos(\psi - \theta_{ij})}{(1 - \rho_{ij}^2)h_{ij}}\right\}}{\pi\Gamma(N)(1 - \rho_{ij}^2)h_{ij}^{N+1}}$$
$$\times K_{N-1}\left(\frac{2N\alpha/h_{ij}}{1 - \rho_{ij}^2}\right), \tag{25}$$

where $\rho_{ij}$ is the magnitude and $\theta_{ij}$ is the phase of the complex coherence factor $\nu_{ij}$, and $h_{ij} = (M_{ii}M_{jj})^{1/2}$. For a sufficiently large number of samples $N$, which is typically greater than ten in practice, the Gaussian asymptote for $S_{ij}[N]$ is achieved. The magnitude $\alpha_{ij}$ obeys the distribution[13]

$$P_{\alpha_{ij}}(\alpha) = \frac{4N^{N+1}\alpha^N}{\Gamma(N)(1 - \rho_{ij}^2)h_{ij}^{N+1}} I_0\left(\frac{2\rho_{ij}N\alpha/h_{ij}}{1 - \rho_{ij}^2}\right)$$
$$\times K_{N-1}\left(\frac{2N\alpha/h_{ij}}{1 - \rho_{ij}^2}\right), \tag{26}$$

which has familiar behavior as the sensors become perfectly correlated, where $\rho_{ij} = 1$, and uncorrelated, where $\rho_{ij} = 0$. Finally, the phase $\psi_{ij}$ is distributed according to[13]

$$P_{\psi_{ij}}(\psi) = \frac{\Gamma(N + 1/2)(1 - \rho_{ij}^2)^N \rho_{ij}\cos(\psi - \theta_{ij})}{2\sqrt{\pi}\Gamma(N)(1 - (\rho_{ij}\cos(\psi - \theta_{ij}))^2)^{N+1/2}}$$
$$+ \frac{(1 - \rho_{ij}^2)^N}{2\pi} F[N, 1; 1/2; (\rho_{ij}\cos(\psi - \theta_{ij}))^2], \tag{27}$$

where $-\pi < \psi_{ij} \leq \pi$. Here standard notation is used for modified Bessel functions $I_0$, $K_{N-1}$, gamma function $\Gamma$, and Gauss's hypergeometric function $F$.

Under the assumption of cross-spectral purity across the array aperture, approximate distributions for the continuous-time sample covariance elements $S_{ij}(T)$ of Eq. (14) are now obtained by substituting the time-bandwidth product $\mu$ of Eqs. (20) and (22) for $N$ in Eqs. (25)–(27). The resulting distributions approximately describe the statistics of finite-time-averaged mutual intensity, and so extend Mandel's ap-

proximate distributions for the diagonal terms $S_{ii}(T)$[11,23] developed in his analysis of photon beam fluctuations.[15] However, barring the trivial case in which the elements are uncorrelated, the joint distribution for all elements of the sample covariance apparently remains unrealized in analytic form, as noted in Sec. I, regardless of whether the sampling is continuous or discrete.

## D. Relationship to broadband performance bounds

Besides Refs. 7–9 there are a number of papers in the literature that examine ocean-acoustic parameter resolution bounds for broadband measurements on a hydrophone array. For example, Refs. 24–26 provide expressions for the corresponding Fisher information matrices in terms of the waveguide Green function. However, the spectral summations and integrals used in these expressions do not include the convoluted effects of finite time windows nor do they include any description of temporal sampling. Exclusion of the former effect limits the validity of these expressions to the asymptotic regime where the measurement's temporal window is so much greater than the coherence time scale that the associated sample size must be statistically large. While no corresponding sample size is explicitly given to scale the Fisher information matrix in these references, this factor may be implicitly defined. Such a scaling is necessary according to Whittle's asymptotic expression for the Fisher information matrix for a set of time series measurements with large sample size.[27] However, a danger in such asymptotic approximations is that their validity may inherently be limited to sample populations of an enormity that cannot be achieved in practice.

While the Fisher information matrices derived in Refs. 28 and 29 also neglect the effects of finite time windowing, they do so by direct use of Whittle's approximate Fisher information matrix and explicitly include the number of independent samples available in the measurement, under the assumption that this number is large. However, the acoustic ray-multipath model of Refs. 28 and 29 cannot be generally reconciled with the full-field propagation models that are presently necessary for realistic ocean-acoustic simulations.[1–9] Therefore the Fisher information matrices derived in Refs. 28 and 29 are of limited practical use in ocean acoustics.

The present analysis stands apart from these studies in that (1) it is not confined to the asymptotic regime but is valid for the general case of $N \geq 1$, and (2) it can be easily implemented with the requisite full-field propagation models that are currently state-of-the-art in ocean acoustics. While the focus is on narrow-band interferometry, all of the expressions derived for the fully randomized field case of Sec. II are directly applicable to broadband problems. For example, Eq. (12) is not limited to narrow-band processes but also defines the Fisher information matrix for joint measurements of a stationary broadband CCGR field, where **K** is the expected covariance of the instantaneous field across the array, $N$ is the number of times an instantaneous field across the array is measured independently, Eqs. (20) and (22) determine the number of such measurements available in a given period, and Eq. (18) gives a spectral representation for the

elements of **K**, where $K_{ij} = M_{ij}$. Finally, it is noteworthy that for sufficiently large $N$, the general expression derived here in Eq. (12) is consistent with Whittle's asymptotic formula.

## III. THE RECEIVED FIELD HAS A DETERMINISTIC COMPONENT

### A. Cramer–Rao bounds for the complex sample covariance

Let the instantaneous field measurements $\phi_i[n]$ contain a deterministic component $\phi_{D_i}[n]$ superposed with a CCGR component $\phi_{G_i}[n]$, such that $\phi_i[n] = \phi_{D_i}[n] + \phi_{G_i}[n]$, which is equivalently described by the vector equation $\mathbf{\Phi}[n] = \mathbf{\Phi}_D[n] + \mathbf{\Phi}_G[n]$. The random part of the field $\phi_{G_i}[n]$ may contain a stochastic signal as well as noise. The deterministic part of the field $\phi_{D_i}[n]$ is assumed to occupy a sufficiently narrow-band about the carrier frequency that $\phi_{D_i}[n]\phi_{D_j}^*[n] \equiv \phi_{D_i}\phi_{D_j}^*$ is time-invariant during collection of the $N$ samples. It is again assumed that the field can be spatially correlated across the array at any instant, and that all field measurements with differing discrete-time indexes are independent.

The expectation value of the sample covariance has elements given by

$$M_{ij} = E\left[ \frac{1}{N} \sum_{n=1}^{N} \phi_i[n]\phi_j^*[n] \right]$$
$$= \phi_{D_i}\phi_{D_j}^* + \langle \phi_{G_i}\phi_{G_j}^* \rangle = \phi_{D_i}\phi_{D_j}^* + M_{G_{ij}}, \quad (28)$$

where each element contains a cross term $\phi_{D_i}\phi_{D_j}^*$ for the deterministic field, and another time-invariant term $M_{G_{ij}}$ for the random field. By use of CCGR moment factoring,[11] it can be shown that the covariance of the sample covariance has elements given by

$$C_{qr,st}(N) = E\left[ \frac{1}{N^2} \sum_{n=1}^{N} \sum_{m=1}^{N} (\phi_q[n]\phi_r^*[n])(\phi_s[m]\phi_t^*[m])^* \right]$$
$$- M_{qr}M_{st}^*$$
$$= \frac{1}{N}(M_{G_{qs}}\phi_{D_r}^*\phi_{D_t} + \phi_{D_q}\phi_{D_s}^*M_{G_{rt}}^* + M_{G_{qs}}M_{G_{rt}}^*), \quad (29a)$$

which becomes

$$C_{qr,st}(N) = \frac{1}{N}(M_{G_{qs}}\phi_{D_r}^*\phi_{D_t} + M_{qs}M_{G_{rt}}^*), \quad (29b)$$

by application of Eq. (28), so that $C_{qr,st}(N) = C_{qr,st}(1)/N$ as anticipated.

Therefore Eq. (10) yields the desired Fisher information matrix upon substitution of the mean defined in Eq. (28), and the covariance defined in Eq. (29a) or (29b), where $[\mathbf{C}^{-1}(N)]_{qr,st} = N[\mathbf{C}^{-1}(1)]_{qr,st}$. Clearly this Fisher information matrix is directly proportional to $N$, and therefore the Cramer–Rao lower bound for any parameter with a finite bound for $N = 1$ must decrease as $N$ increases.

A more compact expression for the Cramer–Rao lower bound

$$E[(\hat{g}_i - g_i)^2] \geq \frac{1}{N} \left[ \left( \frac{\partial \mathbf{M}^T}{\partial \mathbf{g}} \, \mathbf{C}^{-1}(1) \, \frac{\partial \mathbf{M}}{\partial \mathbf{g}} \right)^{-1} \right]_{ii}, \quad (30)$$

is easily obtained from Eqs. (6) and (7). With the definitions $[\mathbf{\Xi}]_{ij} \equiv \phi_{D_i} \phi_{D_j}^*$, and $[\mathbf{M}_G]_{ij} \equiv M_{G_{ij}}$, the covariance of Eq. (29) may be written as

$$\mathbf{C}(N) = \frac{1}{N} \, (\mathbf{M}_G \mathbf{\Xi}^T + \mathbf{M} \mathbf{M}_G^T). \quad (31)$$

This leads to the explicit bound

$$E[(\hat{g}_i - g_i)^2] \geq \frac{1}{N} \left[ \left( \frac{\partial \mathbf{M}^T}{\partial \mathbf{g}} \{ \mathbf{M}_G \mathbf{\Xi}^T + \mathbf{M} \mathbf{M}_G^T \}^{-1} \frac{\partial \mathbf{M}}{\partial \mathbf{g}} \right)^{-1} \right]_{ii}, \quad (32)$$

which may be computationally expensive to implement because it requires two matrix inversions. It is noteworthy that while the matrix $\partial \mathbf{M}/\partial \mathbf{g}$ is not invertible in general, the bound may always be written in the form

$$E[(\hat{g}_i - g_i)^2] \geq \frac{1}{N} \left[ \frac{\partial \mathbf{g}}{\partial \mathbf{M}} \, \mathbf{C}(1) \, \frac{\partial \mathbf{g}^T}{\partial \mathbf{M}} \right]_{ii}, \quad (33)$$

if the matrix $\partial \mathbf{g}/\partial \mathbf{M}$ exists[30] as it must when the problem is properly constrained. More specifically, this bound is given by

$$E[(\hat{g}_i - g_i)^2] \geq \frac{1}{N} \left[ \frac{\partial \mathbf{g}}{\partial \mathbf{M}} \{ \mathbf{M}_G \mathbf{\Xi}^T + \mathbf{M} \mathbf{M}_G^T \} \frac{\partial \mathbf{g}^T}{\partial \mathbf{M}} \right]_{ii}. \quad (34)$$

This form requires no matrix inversion, but it may be less plausible to implement than Eq. (32) because the matrix $\partial \mathbf{g}/\partial \mathbf{M}$ is usually more difficult to determine than $\partial \mathbf{M}/\partial \mathbf{g}$ for applications in ocean-acoustic interferometry.

Due to its temporal invariance over the collection of the $N$ measurement samples, the deterministic component $\mathbf{\Xi}$ is effectively monochromatic. By defining $Q_{G_{ij}}(f)$ as the Fourier transform of the mutual coherence function

$$\langle \phi_{G_i}(t+\tau) \phi_{G_j}^*(t) \rangle,$$

where $\phi_{G_i}(t)$ is a continuous-time measurement of the random field component, the expected covariance $\mathbf{M}_G$ of the random field component can be expressed spectrally as

$$M_{G_{ij}} = \int_{-\infty}^{\infty} Q_{G_{ij}}(f) df, \quad (35)$$

in accord with the analysis of Sec. II B. This representation of $\mathbf{M}_G$ is convenient for numerical implementation of the bound given in Eqs. (32) and (34) because many state-of-the-art ocean-acoustic propagation models operate in the spectral domain.

It is noteworthy that when the deterministic component vanishes, so that $\mathbf{\Xi} = 0$, $\mathbf{M}$ and $\mathbf{M}_G$ are identical. The bound can then also be obtained from the Fisher information matrix of Eq. (12), for the fully randomized field case, and is therefore directly applicable to broadband measurements. Conversely, if the random component of the field vanishes so that $\mathbf{M}_G = 0$, the covariance $\mathbf{C}(N)$ is zero. In this case, the

bound is also zero, corresponding to perfect resolution for any parameters that can be fully described by their dependence on the mean vector $\mathbf{M}$ and known constants.

## B. Sample size as a function of time and coherence

The number of independent samples available in a stationary measurement period is given by the time-bandwidth product of the received field's random component. This number can be obtained by substituting

$$\langle \phi_{G_i}(t+\tau) \phi_{G_i}^*(t) \rangle / M_{G_{ii}}$$

for $\gamma(\tau)$ in Eq. (20), under the assumption that the random component is cross-spectrally pure over the array aperture. As of yet, mutual-intensity distributions analogous to those of Sec. II C have apparently not been derived for the case of a deterministic signal in a CCGR field.

It is significant that the differing assumptions about the received field made in Secs. II and III have lead to differing Fisher information matrices. Clearly, the statistics of the problem at hand must be evaluated before bounds can be assigned. Otherwise, such arbitrary considerations as the mathematical convenience of a particular formulation of the Fisher information begin to motivate the theoretical analysis. It is equally significant that these results are unified by the fact that Fisher information increases linearly with the number of independent samples used to construct the sample covariance, which is a natural consequence of the more general formulation given in Sec. I that follows standard statistical methods.

## IV. ILLUSTRATIVE EXAMPLES

The Fisher information matrix derived in Sec. II can be directly applied to appropriately rescale monochromatic bounds that already exist in the literature. For example, the resolution bounds given in Refs. 1–5 can be made more practical by dividing the limiting minimum mean-square estimation error presented by the appropriate number of independent and stationary samples. This number can be determined by Eq. (20) or (22) if the temporal coherence of the received field and the stationary time of the physical processes causing the fluctuations are known.

Knowledge of the maximum allowable sample size is most critical when monochromatic bounds suggest inadequate resolution for a given parameter. Two narrow-band examples from the literature are presented. In both examples, the entire received field is assumed to obey CCGR statistics, and the original Fisher information matrix from the literature corresponds to that given by Eq. (12) for the single instantaneous measurement case of $N = 1$, where the measurement band about central frequency $f_0$ is sufficiently narrow to approximate the expected instantaneous covariance $\mathbf{K}$ of Eq. (18), where $K_{ij} = M_{ij}$, with a monochromatic model spectrum for which $\mathscr{S}(f) = \delta(f - f_0)$.

## A. Matched-field tomography of internal waves

First consider Ref. 5, where internal-wave parameters are to be estimated by full-field acoustic tomography with a controlled source signal of finite duration and bandwidth. To

resolve a long internal wave to roughly an acoustic wavelength $\lambda_{acou}$, a measurement time $T$ much less than an internal wave period $\tau_{int}$ is generally required, under the assumption that $\lambda_{acou}$ is much less than the internal wavelength. Specifically, the maximum measurement time $T_{max}$ should be no greater than $\lambda_{acou}/c_{int}$. For a typical two-layer shallow water waveguide with density difference to density ratio $\Delta\rho/\rho = 10^{-3}$, upper layer thickness $H = 20$ m, and gravitational acceleration $g$, the long internal wave speed

$$c_{int} = (gH\Delta\rho/\rho)^{1/2} \tag{36}$$

is roughly 0.4 m/s. To obtain more than a single sample, the minimum bandwidth should be greater than $c_{int}/\lambda_{acou}$. The restriction that the processing be narrow band then sets the maximum bandwidth $B_{max}$ and sample size $N_{max} = T_{max}B_{max}$. In narrow-band beamforming, $B_{max}$ must be much less than the propagation speed divided by the array aperture. However, in narrow-band matched-field processing, the propagation range must also be considered to insure the acoustic interference structure on the array does not vary significantly over the measurement band. In this case, the maximum bandwidth must typically be determined by numerical modeling.

For the 500-Hz carrier frequency of Ref. 5, $B_{max}$ must be greater than $1/T_{max} = c_{int}/\lambda_{acou} = 0.13$ Hz to obtain more than a single sample. But parabolic equation modeling shows that $B_{max}$ can be much greater than 0.13 Hz and still be sufficient for narrow-band matched-field inversion over the ranges and waveguide discussed in Ref. 5, so that $N_{max}$ can greatly exceed unity. Therefore, the monochromatic parameter resolution bounds presented in Ref. 5 may be significantly improved by time or frequency averaging. Further, the observation of that reference that certain parameters may not be adequately resolved, even if the number of spatial samples with unique expectation values exceeds the number of parameters to be estimated, is not necessarily true for the specific examples given therein if $N_{max}$ is permitted to be greater than unity in accord with the foregoing analysis.

Finally, it is noteworthy that the CCGR fields assumed for this example may in reality be caused by such diverse mechanisms as high-frequency gravity-wave induced fluctuation of the waveguide's surface boundary or refractive index, motion of the source or receiver, or an incoherent source.

## B. Target detection using ambient noise as a source of opportunity

Now consider Ref. 3, where an object submerged in a waveguide is to be detected using ambient noise as a source of opportunity. In that reference, monochromatic bounds are derived for the range, cross-range and depth resolution of the object. But the limiting resolutions given for cross-range and depth are extremely poor even at very short ranges from the object. In the present context, those mean-square error bounds become more realistic when they are divided by the maximum allowable sample size $N_{max}$. For example, it may be possible to resolve the object if $N_{max}$ is sufficiently large that it equals the corresponding monochromatic bound of Ref. 3 divided by the square of the object scale. However, even within short ranges of a few object lengths, this ratio

requires $N_{max}$ to be greater than $10^4$. When $N_{max}$ is equated with the time-bandwidth product of the received signal, it becomes apparent that no combination of temporal coherence and stationarity for the CCGR ocean surface noise process will make that detection scheme practical beyond a few object lengths. The conclusion may not be the same for highly directional noise, however, as is noted in Ref. 31.

Therefore, it is not only the Fisher information for a monochromatic or instantaneous measurement, but also the coherence and stationary time scales of the random component of the received field that are essential in determining parameter resolution bounds in these examples. It is anticipated that this will be the case in many other situations of practical interest in ocean acoustics, as no doubt many researchers are already aware.

## V. DISCUSSION AND CONCLUSIONS

It is currently common practice in theoretical ocean acoustics to derive parameter resolution bounds that are exclusively for a monochromatic measurement of the fluctuating field received by a hydrophone array. Often the intent of these derivations is to compare the resulting bounds with the resolution of numerical simulations obtained from single-frequency models on a digital computer. It is only natural that such comparisons should turn out favorably. This is because the single-frequency model inherently assumes zero bandwidth which is equivalent to the absence of temporal uncertainty—a perpetually instantaneous field structure. A physically relevant analogy to this situation is the static but random interference pattern or speckle found in the reflection of a coherent laser beam from a smooth surface with wavelength-scale roughness.[32] In both of these cases, the randomness is only exhibited by a spatial sampling. And in both cases the resulting intensity distribution is exponential with a standard deviation equal to the mean, implying a standard deviation for the intensity level equal to Dyer's 5.6 dB,[33] if (1) the fields are sufficiently randomized to be stationary CCGR variables, and (2) the spatial aperture is less than or equal to the speckle scale.

However, once finite bandwidth is introduced by either (1) an incoherent source, (2) relative motion between source and receiver, (3) relative motion between scatterer and source or receiver, (4) fluctuation of the scatterer or waveguide boundary, (5) medium scintillation, or (6) the simple introduction of noise, the randomness takes on a temporal dependence. Averaging the intensity in time at any point in space reduces the variance by the number of independent and stationary fluctuations $N$. Similarly, the intensity-level standard deviation for CCGR fields is approximated by the decaying quantity $(10\log e)/N^{1/2}$ dB with increasing accuracy as $N$ increases.[23,34] In this context, a monochromatic measurement is like an instantaneous one in that it comprises a single sample of a temporally fluctuating quantity. But a single sample constitutes the least favorable sample population as far as parameter estimation is concerned because it yields the largest variance. This is well known in applied ocean acoustics where parameter estimates are typically derived from ensemble averages such as the complex sample

covariance which is widely used in narrow-band matched field processing and beamforming.

To bridge the gap between these current theoretical and applied approaches, the Fisher information for the sample covariance is shown to be equal to the number of independent and stationary samples available times the Fisher information for a single sample. Therefore, *there are no practical limits on parameter resolution* if (1) the bound for a single sample is finite, which is generally the case of interest, (2) a sufficiently large population of independent samples can be found. The parameter resolution issue then becomes one of determining the maximum number of such samples.

A means of determining the number of independent samples available in a given measurement period is presented. Some specific analytical results are given in terms of the temporal coherence of the received field, which simplify greatly under the assumption of cross-spectral purity in spatial measurements across the hydrophone array. This approach has also led to a means of applying recently derived probability distributions for the elements of the discrete sample covariance to approximate similar distributions for finite-time-averaged mutual intensity.

As a final point of discussion, the Cramer–Rao lower bound is typically used to evaluate the performance of parameter estimators. From the perspective of classical estimation theory,[19–22] optimal estimators are unbiased and attain this lower bound. While optimal estimators are only certain to be found if the parameters to be estimated are linearly related to the expectation value of the measurements, maximum likelihood estimators are asymptotically optimal for a sufficiently large number of independent measurement samples even if the relationship between the parameter set and measurement expectation value is nonlinear.[19–22] This last fact is of particular significance in ocean-acoustic interferometry because the nonlinear matched field estimators generally required may only attain optimality in the asymptotic limit of a large sample population. This highlights the significance of not only including the sample size as a variable but also of determining its maximum allowable value when establishing bounds on ocean-acoustic parameter resolution.

## ACKNOWLEDGMENTS

## APPENDIX: KRONECKER PRODUCTS AND THEIR INVERSES

From Ref. 17, the Kronecker product of the double subscripted vectors $\mathbf{A}$ and $\mathbf{B}$ is given by

$$[\mathbf{A}\otimes\mathbf{B}]_{qr,st}\equiv[\mathbf{A}]_{qs}[\mathbf{B}]_{rt}. \qquad (A1)$$

When the inverses $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ exist, then

$$(\mathbf{A}\otimes\mathbf{B})^{-1}=\mathbf{A}^{-1}\otimes\mathbf{B}^{-1}, \qquad (A2)$$

so that

$$[(\mathbf{A}\otimes\mathbf{B})^{-1}]_{qr,st}=[\mathbf{A}^{-1}]_{qs}[\mathbf{B}^{-1}]_{rt}. \qquad (A3)$$

Therefore, Eq. (9) indicates that

$$\mathbf{C}(N)=\frac{1}{N}\,\mathbf{M}\otimes\mathbf{M}^*. \qquad (A4)$$

Application of Eq. (A3) then yields

$$[\mathbf{C}^{-1}(N)]_{qr,st}=\frac{1}{N}\,[\mathbf{M}^{-1}]_{qs}[\mathbf{M}^{*-1}]_{rt}, \qquad (A5)$$

under the assumptions of Sec. II A.

For notational convenience, the definition $\mathbf{AB}\equiv\mathbf{A}\otimes\mathbf{B}$ is used for the Kronecker product in the main text.

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2860   J. Acoust. Soc. Am., Vol. 99, No. 5, May 1996

Nicholas C. Makris: Parameter resolution bounds   2860

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[31] However, the situation is entirely different for range estimation when the direct noise is correlated with the scattered noise. In this case, the correlation enables a coherent gain within the signal band by matched filtering with the dominant noise signal. (Here the dominant noise signal is defined as that which dominates the scattered field received from the object.) This is exhibited in the monochromatic bounds for range estimation given in Ref. 3. In homogeneous surface noise, the dominant noise signal is in the forward direction[3] so that separation of direct and scattered noise is impractical. However, the matched-filter approach may be practical in directional noise near the coast if the direction of the dominant noise signal can be well separated from the scattered noise. This concept is presently being investigated in the field and with simulations.

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